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The nucleolus of homogeneous games with steps

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Abstract

Homogeneous games were introduced by von Neumann and Morgenstern in the constant-sum case. Peleg studied the kernel and the nucleolus within this framework. However, for the general nonconstant-sum case Ostmann invented the unique minimal representation, Rosenmüller gave a second characterization and Sudhölter discovered the "incidence vector". Based on these results Peleg and Rosenmüller treated several solution concepts for "games without steps". The present paper treats the case of games "with steps". It is shown that with a suitable version of a "truncated game" the nucleolus of a game is essentially the one obtained by truncating behind the "largest step". As the truncated version has "no steps", the case "with steps" is reduced to the one "without steps", which is treated in the paper by Peleg and Rosenmüller.

Homogeneous games

The material of this paper is organized as follows. This section serves as an introduction to the theory of homogeneous games and provides the necessary concepts and notations. Section 1 deals with certain families of representations of homogeneous games "with steps". These families put an increasing amount of weight at the players within the lexicographically first minimal winning coalition (Theorem 1.11). As a consequence, it turns out that in games with steps, the system of minimal winning coalitions cannot be (weakly) balanced (Corollary 2.6). This is of course important in context with the structure of the nucleolus; thus Section 2 discusses some simple properties of the nucleolus. However, Corollary 2.6 is not sufficient to explain the structure of the nucleolus of a homogeneous game "with steps". Therefore, Section 3 explains the "reduction theory" of the nucleolus. First, a truncation procedure is necessary. This, after some preliminary work, is described by Definition 3.6. Lemma 3.7 explains the nature of the truncated game. Finally, by

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Theorem 3.8 and Corollary 3.9 we collect the material available so far and prove that the nucleolus of a game with steps reduces to the one of the truncated version.

This section serves as an introduction to the theory of homogeneous (simple) games. All of the material presented may be found in the literature, see e.g. Ostmann [6], Rosenmüller [10], Sudhölter [14].

Let $\mathbb{N} = \{1,2,3,\ldots\}$ denote the "universe of *players*". For the "grand coalition" we choose some "interval" $\Omega = [a,b] = \{i \in \mathbb{N} \mid a \leqslant i \leqslant b\}$. $\underline{\underline{P}} = \underline{\underline{P}}(\Omega) = \underline{\underline{P}}([a,b]) = \{S \mid S \subseteq \Omega\}$ is the system of *coalitions*. If

$$v : \underline{\underline{P}} \to \mathbb{R}, \quad v(\emptyset) = 0$$

is a mapping on $\underline{\underline{P}}$, then $(\Omega, \underline{\underline{P}}, v)$ is a *game*; somewhat sloppily we refer to v as to "a game". v is *simple* if $v : \underline{\underline{P}} \to \{0,1\}$ holds true.

Unions of coalitions and players are written $S \cup i$ instead of $S \cup \{i\}$; $S + T$ and $S + i$ denote disjoint unions. Similarly, $i < T$ denotes $i < j$ $(j \in T)$ $(S, T \in \underline{\underline{P}}, i, j \in \Omega)$.

Given a simple game v, $\underline{\underline{W}} = \underline{\underline{W}}(v) = \{S \in \underline{\underline{P}} \mid v(S) = 1\}$ is the system of *winning* coalitions while

$$\underline{\underline{W}}^m = \underline{\underline{W}}^m(v) = \{S \in \underline{\underline{W}} \mid v(T) = 0 \text{ for } T \subsetneq S\}$$

is the system of *minimal* winning coalitions ("min-win coalitions").

A vector $M = (M_i)_{i \in \Omega} \in \mathbb{R}_+^\Omega$ is tantamount to a function on $\underline{\underline{P}}$ via $M(S) = \sum_{i \in S} M_i$ $(S \in \underline{\underline{P}})$ (thus, it is a nonsimple "game") and hence called a "*measure*" (M is additive). Games and in particular measures, may be restricted on subsets $T \subseteq \Omega$, the notation is $v|_T$ or $M|_T$; e.g.

$$v|_T(S) = v(T \cap S) \quad (S \in \underline{\underline{P}}),$$

$$v|_T(S) = v(S) \qquad (S \in \underline{\underline{P}}(T));$$

the version living on $\underline{\underline{P}}(\Omega)$ and the one living on $\underline{\underline{P}}(T)$ are not distinguished. We tolerate $v|_\emptyset$.

If M is a measure and $\lambda > 0$, then (M, λ) is a *representation* of v if

$$v(S) = \begin{cases} 1, & M(S) \geqslant \lambda, \\ 0, & M(S) < \lambda \end{cases}$$

holds true, in this case we write $v = v_\lambda^M$. Of course, integer representations are of particular interest.

A measure M is said to be *homogeneous* w.r.t. $\lambda \in \mathbb{R}_{++}$ (written "$M \hom \lambda$") if, for any $T \in \underline{\underline{P}}$ with $M(T) > \lambda$, there is $S \subseteq T$ with $M(S) = \lambda$.

A game v is *homogeneous* if there exists a representation (M, λ) with $M \hom \lambda$ and $v(\Omega) = 1$. (The definition is due to von Neumann and Morgenstern [15].)

We assume all representable games to be *directed*, i.e., there exists a representation (M, λ) such that $i < j$ implies $M_i \geqslant M_j$ $(i, j \in \Omega)$.

Thus, the "strong" or "large" players (the ones with big weight M_i) are first in enumeration (or "index"). In particular, if $l(S) = \max\{i \mid i \in S\}$ for $S \in \underline{\underline{P}}$, then $l(S)$ is the "weakest", "smallest", or last player in S.

While players are ordered according to "size", coalitions are ordered lexicographically. In particular, the lex-max min-win coalition is the lexicographically first minimal winning coalition; in a homogeneous game with a homogeneous representation (M, λ) this coalition is sometimes denoted by $S^{(0)}$ or $S^{(\lambda)}$ (an interval with measure $M(S^{(\lambda)}) = \lambda$).

Players i and j are of the same *type* (written $i \sim j$ or $i \sim_v j$), if, for all $S \subseteq \Omega - (i + j)$, $v(S + i) = v(S + j)$ holds true. A representation (M, λ) is *symmetric* if $i \sim j$ implies $M_i = M_j$ $(i, j \in \Omega)$.

Player $i \in \Omega$ is a *dummy* if $v(S \cup i) = v(S)$ for all $S \in \underline{\underline{P}}$. All dummy players are of the same type. Note that the game is assumed to be directed; thus, the definition of types induces a decomposition of Ω into *intervals*

$$\Omega = T_1 + \cdots + T_r$$

of players of one type. $i \in T_\rho$ is also expressed as "i is of type ρ"; thus

$$R = \{1, \ldots, r\}$$

denotes the set of types. If dummies are present, then r is "the dummy type" and, in a natural way, type ρ is "stronger" than type $\rho + 1$ $(\rho \in R - r)$.

We shall refer to "dummy" as to a *character* that may or not be attached to a player. There are two further characters, "sum" and "step", which we are going to explain now.

To this end, fix a nondummy player $i \in \Omega$.

Among all min-win coalitions containing i, let $L^{(i)}$ be a one with minimal length, i.e.,

$$l(L^{(i)}) = \min\{l(S) \mid S \ni i,\ S \in \underline{\underline{W}}^m\}.$$

Then

$$C^{(i)} := [l(L^{(i)}) + 1, b]$$

is the *domain of* i and $M^{(i)} := M|_{C^{(i)}}$ is i's *satellite measure*.

Now, if $M^{(i)}(C^{(i)}) \geqslant M_i$, then i is a *sum* ("his character is sum"), since he may be replaced in a min-win coalition by a coalition of smaller players, his weight being the sum of the weights of the smaller players.

In this case, we call $v^{(i)} := v_{M_i}^{M^{(i)}}$ the *satellite game* of i (a homogeneous game!). Also, $S^{(i)} = S_{M_i}^{M^{(i)}}$ is the coalition of i's *satellites*; this is the lex-max min-win coalition of $v^{(i)}$.

Otherwise, if $M^{(i)}(C^{(i)}) < M_i$, then i is a *step*. In this case his ("pseudo") satellites are the members of his domain, i.e., we put $S^{(i)} := C^{(i)}$.

"Sum" and "step" are possible characters of a player – like dummy. From this, there results a further decomposition of Ω into the sets of characters

$$\Omega = \Sigma + \Pi + \Delta,$$

where $\Sigma = \Sigma(v) = \{i \in \Omega \mid \text{is a sum}\}$, $\Pi = \Pi(v) = \{\text{steps}\}$ and $\Delta = \Delta(v) = \{\text{dummies}\}$. Σ as well as Δ may be empty while Π is not.

Remark 0.1. The following is well known (Ostmann [6], Rosenmüller [10], Sudhölter [14]).

(1) The smallest nondummy player is always a step. Its domain may be the empty set. If v is a constant-sum game, then the smallest nondummy is the only step.

(2) A homogeneous game has a unique minimal representation $(\bar{M}, \bar{\lambda})$ (e.g., in the sense that $(\bar{M}, \bar{\lambda})$ is integer and $\bar{M}(\Omega)$ is minimal), this representation is symmetric and attaches weight 0 to dummies.

(3) A pair (M, λ) is a homogeneous representation of v iff there exists real numbers $\Delta_i \geqslant 0$ with $\Delta_i > 0$ $(i \in \Pi)$, $\Delta_i = 0$ $(i \in \Sigma)$ such that

$$M_i = \Delta_i \qquad (i \in \Delta),$$
$$M_i = \Delta_i + M(S^{(i)}) \quad (i \in \Sigma \cup \Pi)$$

holds true. (Δ_i $(i \in \Omega)$ is the "jump at i".) The unique minimal representation is obtained by putting $\Delta_i = 0$ $(i \in \Delta)$, $\Delta_i = 1$ $(i \in \Pi)$.

(4) Let $j \in \Sigma$ and let $i \in C^{(j)}$ be a nondummy w.r.t. $v^{(j)}$ (suitably, we write $i \notin \Delta^{(j)} := \Delta(v^{(j)})$). Then i has domain, satellites etc. w.r.t. $v^{(j)}$; let $C^{(i,j)}$, $M^{(i,j)}$ denote his domain and satellite measure w.r.t. $v^{(j)}$. Then

$$C^{(i)} = \bigcup \{C^{(i,j)} \mid j \in \Sigma,\ i \in C^{(j)},\ i \notin \Delta^{(j)}\},$$
$$M^{(i)} = \max \{M^{(i,j)} \mid j \in \Sigma,\ i \in C^{(j)},\ i \notin \Delta^{(j)}\}$$

holds true with an obvious interpretation of "max".

(5) If $i \notin S^{(\lambda)}$, then $i \in \Sigma$ iff $i \in \Sigma^{(j)}$ for some $j \in S^{(\lambda)}$. Also, $i \in \Delta$ if $i \in \Delta^{(j)}$ for all $j \in S^{(\lambda)}$.

(6) Let $j \in S^{(\lambda)}$ and let $l^\lambda = l(S^\lambda)$. Then $C^{(j)} = [l^\lambda + 1, b]$. $S^{(\lambda)} \cap \Pi$ is the coalition of "inevitable players" (i.e., those that are present in every min-win coalition). If all players in $S^{(\lambda)}$ are steps (inevitable), then v is the unanimous game of the members of S^λ (with minimal representation

$$(\bar{M}, \bar{\lambda}) = (\underbrace{1, \ldots, 1}_{\lambda}, 0, \ldots, 0; \bar{\lambda})).$$

Apart from the inevitable players, no further steps occur in S^λ.

(7) In every homogeneous representation (M, λ) of v, sums of the same type have the same weight. Steps of the same type may have different weight, but then they appear or do not appear simultaneously ("as a block") in every min-win coalition.

1. Monotone representation

Representations of homogeneous games are essentially defined by prescribing the "jumps" at the various steps. This section serves to study the consequences if the jumps are considered to be (positive) affine functions of a real parameter.

Lemma 1.1 (small steps belong to i's domain). *Let v be a homogeneous game and let $i < \tau$ be two players of different type. If $\tau \notin \Sigma$, then $\tau \in C^{(i)}$.*

Proof. By induction on the number of types. If there is just one type, nothing has to be proved.

Otherwise, let $S^{(\lambda)}$ be the lex-max min-win coalition. τ cannot be an inevitable player (since i precedes him and is of different type). Also, τ cannot be one of the sums in $S^{(\lambda)}$. Hence, $\tau \notin S^{(\lambda)}$, and if $i \in S^{(\lambda)}$ the proof is done.

If $i \notin S^{(\lambda)}$, then consider, for every sum $j \in S^{(\lambda)}$ the satellite game $v^{(j)}$. In at least one of these satellite games, i and τ are of different type. Since $\tau \notin \Sigma$, we have $\tau \notin \Sigma^{(j)}$. By induction, $\tau \in C^{(i,j)}$. Since

$$C^{(i)} = \bigcup_j C^{(i,j)},$$

our claim follows. □

In order to simplify matters we will, for the remainder of this section, assume that all games under consideration have no dummies.

Definition 1.2. Let $i, \tau \in \Omega$ and $i < \tau \in \Pi$. We shall say that τ is the *next step following* i if i and τ are of different type and there is no step in $[i+1, \tau-1]$.

Before proceeding with representation theory we have to shortly discuss two mechanisms connected with the replacement of players (sums) by smaller players in a homogeneous game.

To this end, let v be a homogeneous game.

Consider $S \in \underline{\underline{W}}^m$ and let $l = l(S)$ again denote the last player in S. Suppose $j \in S$ is such that

$$[j, l] \subseteq S, \quad S - j + [l+1, n] \in \underline{\underline{W}}. \tag{1}$$

Then j is *expendable*; we may replace him in S by an interval of smaller players, thus generating a coalition

$$\rho_j(S) := S - j + [l(S)+1, t] \tag{2}$$

where t is uniquely defined by $M([l(S)+1, t]) = M_j$. This procedure is based on the Basic Lemma (Rosenmüller [10]), see Sudhölter [14].

On the other hand, let $T \in \underline{\underline{W}}^m$ and suppose that $r \notin T$ satisfies

$$[r+1, l(T)] \subseteq T. \tag{3}$$

Then r is the *last dropout* (see Rosenmüller [10]) and there is a unique $t' \in [r+1, l(T)]$ such that

$$\varphi(T) := T + r - [t', l(T)] \tag{4}$$

is min-win. That is, φ inserts the last dropout and cuts off an appropriate tail of T as to generate a min-win coalition. And thus, ρ_j renders j to be the last dropout if he is expendable in S.

Clearly, if r is the last dropout in T, then (he is expendable in $\varphi(T)$ and)

$$\rho_r(\varphi(T)) = T. \tag{5}$$

Similarly, if j is expendable in S then (he is the last dropout in $\rho_j(S)$ and)

$$\varphi(\rho_j(S)) = S \tag{6}$$

holds true.

Definition 1.3. Let v be a homogeneous game. A family of representations

$$(M(a), \lambda(a))_{a \in \mathbb{R}_{++}}$$

is said to be *affine*, if there are constants $A_i \geqslant 0$ and $B_i > 0$ ($i \in \Pi$) such that $M_i = M_i(a)$ satisfies

$$M_i = A_i a + B_i + M^{(i)}(S^{(i)}) \quad (i \in \Pi), \tag{7}$$

$$M_i = M^{(i)}(S^{(i)}) \quad (i \in \Sigma), \tag{8}$$

$$\lambda(a) = M(S^0) \quad \text{(where } S^0 \text{ is the lex-max min-win coalition)} \tag{9}$$

that is, the jump at every step is a ("positive") affine function on $\mathbb{R}_{++}$.

Remark 1.4. (1) For $a \in \mathbb{R}_{++}$, $(M(a), \lambda(a))$ is a homogeneous representation.

(2) By induction it is easily seen that there are vectors E, $F \in \mathbb{R}_+^\Omega$ such that $F > 0$ and

$$M_i = M_i(a) = E_i a + F_i \quad (i \in \Omega), \tag{10}$$

hence

$$M(a)(S) = aE(S) + F(S) \quad (S \in \underline{\underline{P}}). \tag{11}$$

Thus, an affine family of representations is equivalent to an affine mapping from $\mathbb{R}_{++}$ into the homogeneous representations of v (regarded as a subset of $\mathbb{R}^\Omega$).

For short, we shall write $(M(\cdot), \lambda(\cdot))$ to indicate an affine family of representations (an a.f.r.).

(3) Satellite measures, lex-max coalitions etc. do not depend on a. Therefore, it makes sense to state that, for each $i \in \Sigma$,

$$(M^{(i)}(\cdot), g_i(\cdot))$$

is an a.f.r. of the satellite game $v^{(i)}$ etc.

Definition 1.5. Let v be homogeneous. An a.f.r. $(M(\cdot), \lambda(\cdot))$ is said to be *monotone* if, for every $i \in \Pi$, the constants A_i and B_i as required by (7) satisfy

$$\frac{A_i}{B_i} \geqslant \frac{E_j}{F_j} \quad (j \in C^{(i)}). \tag{12}$$

A monotone $(M(\cdot), \lambda(\cdot))$ is *strictly monotone* "*at* $i \in \Pi$" if $C^{(i)} \neq \emptyset$ and (12) is a strict inequality for some $j \in C^{(i)}$, thus i is not of the smallest step's type by Remark 1.7.

A monotone $(M(\cdot), \lambda(\cdot))$ is *strictly monotone* if it is strictly monotone at some $i \in \Pi$ which is automatically *not of the type of the smallest step.*

Note 1.6. For nonnegative reals a, b, c, d with $b, d > 0$ it is clear that

$$\max\left(\frac{a}{b}, \frac{c}{d}\right) \geqslant \frac{a+c}{b+d} \geqslant \min\left(\frac{a}{b}, \frac{c}{d}\right) \tag{13}$$

and, in addition, if one inequality of (13) is strict, then so is the other one.

Remark 1.7. Suppose v has two steps of different types, that is, there is $i \in \Pi$ such that i and b are of different type. In view of Lemma 1.1, we know that $C^{(i)} \neq \emptyset$. It is then easily established that there *exists* a strictly monotone affine family of representations of v.

Lemma 1.8. *Let v be a homogeneous game and let $(M(\cdot), \lambda(\cdot))$ be a monotone a.f.r. Let $i \in \Omega$ be such that $C^{(i)} \neq \emptyset$. Then the following two statements hold true:*

(1)

$$\frac{E_i}{F_i} \geqslant \frac{E(C^{(i)})}{F(C^{(i)})}. \tag{14}$$

(2) *For all $i, j \in \Omega$ with $i \leqslant j$, $C^{(i)} \supseteq C^{(j)} \neq \emptyset$,*

$$\frac{E(C^{(i)})}{F(C^{(i)})} \geqslant \frac{E(C^{(j)})}{F(C^{(j)})}. \tag{15}$$

Proof. We proceed by backwards induction beginning with $i = b$. If $i = b$, then $C^{(i)} = \emptyset$ and nothing has to be shown.

Next, pick $i_0 \in [1, b-1]$ and assume that the lemma is true for all $i \in [i_0 + 1, b]$.

Again, if $C^{(i_0)} = \emptyset$, then we have nothing to prove. Therefore we assume $C^{(i_0)} \neq \emptyset$ and we proceed by verifying (14).

Naturally, we distinguish two cases according to whether i_0 is step or sum.

Case 1: *i_0 is step.* Then

$$\frac{A_{i_0}}{B_{i_0}} \geqslant \frac{E(C^{(i_0)})}{F(C^{(i_0)})}$$

(by (12), (13) and (14)), thus

$$\frac{E_{i_0}}{F_{i_0}} = \frac{A_{i_0} + E(C^{(i_0)})}{B_{i_0} + E(C^{(i_0)})} \geqslant \frac{E(C^{(i_0)})}{F(C^{(i_0)})}.$$

Case 2: *i_0 is sum.* Then $E_{i_0} = E(S^{(i_0)})$ and $F_{i_0} = F(S^{(i_0)})$ by the recursive definition of weights (cf. Definition 1.3 and Remark 1.4).

If it so happens that $S^{(i_0)} = C^{(i_0)}$, the proof is already finished. Otherwise, we proceed as follows using the induction hypothesis. We have, for all $i \in S^{(i_0)}$ and $i_1 = l(S^{(i_0)})$,

$$\frac{E_i}{F_i} \geqslant \frac{E(C^{(i)})}{F(C^{(i)})} \geqslant \frac{E(C^{(i_1)})}{F(C^{(i_1)})},$$

and hence

$$\frac{E(S^{(i_0)})}{F(S^{(i_0)})} \geqslant \frac{E(C^{(i_1)})}{F(C^{(i_1)})} \quad \text{by (13)}$$

as well as

$$\frac{E_{i_0}}{F_{i_0}} \geqslant \frac{E(S^{(i_0)}) + E(C^{(i_1)})}{F(S^{(i_0)}) + F(C^{(i_1)})} = \frac{E(C^{(i_0)})}{F(C^{(i_0)})};$$

this completes the induction to verify (14).

Now, in order to verify (15), consider some j, $i_0 < j$, such that

$$\emptyset \neq C^{(j)} \subsetneqq C^{(i_0)}.$$

It suffices to show (15) for such a j that yields a maximal size $|C^{(j)}|$.

By definition we can find a coalition $S \in \underline{\underline{W}}^{(m)}(v)$ such that

$$l(S) = \min C^{(j)} - 1.$$

We claim that $[\min C^{(io)}, l(S)] \subseteq S$, indeed, otherwise $\min C^{(io)} < l(\varphi(S)) < l(S)$ would contradict the maximality of $|C^{(j)}|$. Thus $C^{(i)} \neq \emptyset$ $(i \in [\min C^{(io)}, l(S)])$.

Using (14) and the induction hypothesis, we have

$$\frac{E_i}{F_i} \geqslant \frac{E(C^{(j)})}{F(C^{(j)})}$$

for $i \in [\min C^{(io)}, l(S)]$ and hence

$$\frac{E(C^{(io)})}{F(C^{(io)})} \geqslant \frac{E(C^{(j)})}{F(C^{(j)})}. \qquad \square$$

A monotone a.f.r. enjoys a certain monotonicity property with respect to the quotient E_i/F_i $(i \in \Omega)$: *essentially* the quotients are increasing with weight (i.e., from right to left).

More precisely let

$$\{l_1, \ldots, l_r\} = \{l(S) \mid S \in \underline{\underline{W}}^m(v)\}$$

denote the lengths of min-win coalitions, ordered with increasing index, i.e.,

$$l_1 < \cdots < l_r = b,$$

(player b is no dummy!).

Then, in view of our definition of $C^{(i)}$, we have

$$\{C^{(i)} \mid i \in \Omega\} = \{[l_j + 1, b] \mid j \in [1, r]\}.$$

Define

$$G_j := \frac{E([l_j + 1, b])}{F([l_j + 1, b])} \quad (j \in [1, r - 1]),$$

then (15) implies that G_j is decreasing in j (i.e., "increasing from right to left"). Also, (14) tells us that

$$\frac{E_i}{F_i} \geqslant G_j \quad (i \in [a, l_j]).$$

Thus, we imagine Fig. 2.

Definition 1.9. Let v be a homogeneous game with steps of different types and let $(M(\cdot), \lambda(\cdot))$ be a *strictly* monotone a.f.r. A coalition $S \in \underline{\underline{W}}^m(v)$ is said to be *strictly monotone*, if $l(S) < b$ and there is a player $i \in S$ with $[i, l(S)] \subseteq S$ such that

$$\frac{E_i}{F_i} > \frac{E([l(S) + 1, b])}{F([l(S) + 1, b])}. \tag{16}$$

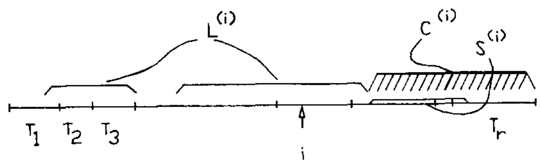

Fig. 1.

Corollary 1.10. *If v is a homogeneous game with steps of different types and $(M(\cdot), \lambda(\cdot))$ is a strictly monotone a.f.r., then the lex-max min-win coalition $S^{(\lambda)}$ is strictly monotone.*

Proof. Definitions 1.5 and 1.9 imply the existence of a strictly monotone coalition S, take $L^{(\tau)}$ if τ induces a strict inequality in (12) (cf. Fig. 1).

Therefore it suffices to show that for all strictly monotone $S \neq S^{(\lambda)}$, it is always true that $\varphi(S)$ is strictly monotone.

To this end pick a strictly monotone $S \neq S^{\lambda}$ and pick $k \in S$ such that $\varphi(S) = (S + k) - [l(\varphi(S)) + 1, l(S)]$. Also pick $i_0 \in S$ satisfying (16) for $i = i_0$.

Clearly, we have to distinguish two cases according to whether i_0 has been dropped by applying φ or not.

Case 1: $i_0 \in \varphi(S)$. Put

$$S^1 = [l(\varphi(S)) + 1, l(\rho_{i_0}(\varphi(S)))]$$

and

$$S^2 = [l(S^1) + 1, l(S)]$$

as well as

$$T = [l(S) + 1, b].$$

Observe that $E(S^1) = E_{i_0}$, $F(S^1) = F_{i_0}$, thus by definition

$$\frac{E(S^1)}{F(S^1)} = \frac{E_{i_0}}{F_{i_0}} > \frac{E(T)}{F(T)}.$$

By (15) and (13),

$$\frac{E(S^2)}{F(S^2)} \geqslant \frac{E(T)}{F(T)}$$

(cf. Fig. 2), thus

$$\frac{E(S^1) + E(S^2)}{F(S^1) + F(S^2)} > \frac{E(T)}{F(T)}$$

and

$$\frac{E(S^1) + E(S^2)}{F(S^1) + F(S^2)} > \frac{E(S^1) + E(S^2) + E(T)}{F(S^1) + F(S^2) + E(T)}$$

$$= \frac{E([l(\varphi(S)) + 1, b])}{F([l(\varphi(S)) + 1, b])}.$$

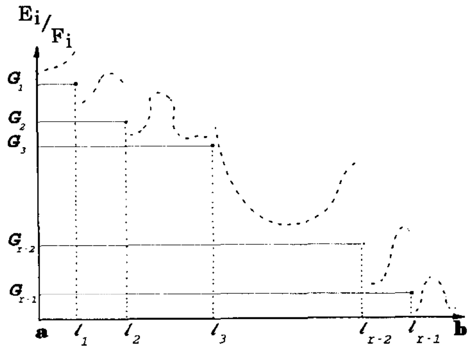

Fig. 2.

This implies that i_0 satisfies (16), suitably changed to i_0 and $\varphi(S)$.

Case 2: $i_0 \notin \varphi(S)$. Put $S^1 = [l(\varphi(S)) + 1, l(S)]$, thus $i_0 \in S^1$ and $E_{i_0}/F_{i_0} > E(T)/E(T)$ where T is defined as in the first step. Using (13) we obtain:

$$\frac{E(S^1)}{F(S^1)} > \frac{E(T)}{F(T)}$$

and

$$\frac{E(S^1)}{F(S^1)} > \frac{E(S^1) + E(T)}{F(S^1) + E(T)}.$$

Again by (15) (see Fig. 2) our claim follows. □

Theorem 1.11. *Let v be homogeneous and $(M(\cdot), \lambda(\cdot))$ an a.f.r. Define*

$$Q: \mathbb{R}_{++} \to \mathbb{R},$$
$$Q(a) = \frac{\lambda(a)}{M(a)(\Omega)}. \tag{17}$$

(1) *If $(M(\cdot), \lambda(\cdot))$ is monotone, then Q is a monotone increasing function in a.*
(2) *If $(M(\cdot), \lambda(\cdot))$ is strictly monotone, then so is Q.*

Proof. Clearly (omitting the argument a)

$$Q(a) = \frac{M(S^{(\lambda)})}{M(\Omega)} = \frac{aE(S^{(\lambda)}) + F(S^{(\lambda)})}{aE(\Omega) + F(\Omega)} \tag{18}$$

In order to show that this is (strictly) monotone, it suffices to show that

$$\frac{E(S^{(\lambda)})}{F(S^{(\lambda)})} \underset{(-)}{>} \frac{E(\Omega)}{F(\Omega)} \tag{19}$$

holds true.

But this inequality is a direct consequence of Lemma 1.8, (13) and (15). For the strict inequality we have in addition to invoke Corollary 1.10. □

2. The nucleolus: preliminary results

From now on we shall always assume that any homogeneous game under consideration has no dummies.

Therefore, the smallest player is always a step (Remark 0.1(1)). Following the tradition of Sudhölter [14] and Peleg and Rosenmüller [9] we speak of a homogeneous game "without steps" if the smallest player is the only step. Note that for games "without steps" the representation is unique up to a multiple and that constant-sum games are games "without steps".

Definition 2.1.

$$\mathscr{X}^* = \mathscr{X}^*(\Omega) = \{x \in \mathbb{R}^\Omega \mid x(\Omega) = 1\} \tag{1}$$

is the set of *pre-imputations*. Also

$$\mathscr{X} = \mathscr{X}(\Omega) = \{x \in \mathscr{X}^*(\Omega) \mid x \geqslant 0\} \tag{2}$$

is the set of *pseudo-imputations*.

The *nucleolus* of a game was introduced by Schmeidler [12], see also Maschler, Peleg and Shapley [5]; usually, it is defined with respect to a set of payoff vectors. Tentatively, the *pre-nucleolus* $\mathscr{N}^*(v)$ is meant to be the one defined with respect to $\mathscr{X}^*$ and the *pseudo-nucleolus* $\mathscr{N}(v)$ is meant to be defined with respect to $\mathscr{X}$.

In [12] it is shown that the pseudo-nucleolus consists of a unique pseudo-imputation $\nu = \nu(v)$ (also called "the pseudo-nucleolus of v").

Note that in our context of homogeneous games, we assume neither superadditivity of v nor do we exclude singletons to be winning coalitions. However, even if single players form winning coalitions ("are winning"), we do not encounter additional problems, for $\mathscr{N}^*$ and $\mathscr{N}$ are equal, as is stated by the following lemma.

For any game v on Ω and $x \in \mathbb{R}^\Omega$ let us use the notation $e(S, x) = v(S) - x(S)$ to denote the *excess* of x (at S). Also let

$$\mu = \mu(x, v) = \max\{e(S, x) \mid S \in \underline{\underline{P}}\} \tag{3}$$

and

$$\mathscr{M} = \mathscr{M}(x, v) = \{S \in \underline{\underline{P}} \mid e(S, x) = \mu(x, v)\}. \tag{4}$$

Now, we have

Lemma 2.2. *Let v be a homogeneous game. Then $\mathscr{N}^*(v) = \mathscr{N}(v)$.*

Proof. Since $\mathscr{X} \subseteq \mathscr{X}^*$, it suffices, given any $x^* \in \mathscr{X}^*$ with negative coordinates, to construct $x \in \mathscr{X}$ such that

$$\mu(x, v) < \mu(x^*, v) \tag{5}$$

holds true.

To this end, fix $x^* \in \mathscr{X}^*$ and define

$$P := \{i \in \Omega \mid x_i^* > 0\}, \qquad N := \{i \in \Omega \mid x_i^* < 0\}, \tag{6}$$

we assume $N \neq \emptyset$. Pick $x \in \mathscr{X}$ such that the following conditions are satisfied

$$0 \leqslant x_i < x_i^* \quad (i \in P), \tag{7}$$

$$0 = x_i \quad (i \in \Omega - P). \tag{8}$$

Now consider $S \in \mathscr{M}(x, v)$. If $P \subseteq S$, the $0 \geqslant e(S,x) = \mu(x,v)$ is verified at once. As $e(\{i\}, x^*) > 0$ $(i \in N)$, we are done, since (5) holds obviously true.

If, on the other hand $P \subseteq\!\!| S$ prevails, then let $S^- := S \cup N$. Clearly

$$v(S^-) \geqslant v(S) \tag{9}$$

since any homogeneous game is monotone (actually, monotonicity is sufficient!). Moreover

$$\begin{aligned} x^*(S^-) &= x^*(S \cap P) + x^*(N) \\ &= x^*(S \cap P) + x(P) - x^*(P) \end{aligned}$$

(since $x^*(N) + x^*(P) = 1 = x(P)$), and thus

$$\begin{aligned} x^*(S^-) &= x(P) - x^*(P - S) \\ &= (x - x^*)(P - S) + x(P \cap S) \\ &= (x - x^*)(P - S) + x(S) \\ &< x(S) \end{aligned} \tag{10}$$

(observe (7) and (8)). Combining (9) and (10) we obtain

$$\begin{aligned} e(S^-, x^*) &= v(S^-) - x^*(S^-) \\ &\geqslant v(S) - x^*(S^-) \\ &> v(S) - x(S) = e(S, x), \end{aligned}$$

which proves (5). □

According to Kohlberg [4], a collection

$$\mathscr{B} = \{\underline{\underline{B}}_0, \ldots, \underline{\underline{B}}_p\}, \quad \underline{\underline{B}}_q \subseteq \underline{\underline{P}} \ (q = 0, \ldots, p)$$

of systems of coalitions is called a *coalition array* if $\underline{\underline{B}}_0$ contains only singletons and

$$\underline{\underline{B}}_1 + \cdots + \underline{\underline{B}}_p = \underline{\underline{P}}$$

holds true.

Given a homogeneous game v on Ω and a pseudo-imputation $x \in \mathscr{X}$, a coalition array $\mathscr{B}(x, v)$, i.e.,

$$\underline{\underline{B}}_0 = \underline{\underline{B}}_0(x), \underline{\underline{B}}_1 = \underline{\underline{B}}_1(x, v), \ldots, \underline{\underline{B}}_p = \underline{\underline{B}}_p(x, v)$$

is specified as follows:

$$\underline{\underline{B}}_0(x) = \{\{i\} \mid x_i = 0\}, \tag{11.1}$$

$$e(S, x) = \text{const} \quad (S \in \underline{\underline{B}}_j(x, v)) \ (j = 1, \ldots, p), \tag{11.2}$$

$$e(S, x) < e(T, x) \quad (S \in \underline{\underline{B}}_j(x, v),\ T \in \underline{\underline{B}}_{j-1}(x, v)) \ (j = 2, \ldots, p). \tag{11.3}$$

A coalition array has *property* I if, for all $q \in [1, \dots, p]$ and $y \in \mathbb{R}^{\Omega}$ satisfying

$$y(S) \geqslant 0 \quad \left(S \in \bigcup_{j=0}^{q} \underline{\underline{B}}_j\right), \tag{12}$$

$$y(\Omega) = 0, \tag{13}$$

it follows that

$$y(S) = 0 \quad \left(S \in \sum_{j=1}^{q} \underline{\underline{B}}_j\right)$$

holds true.

A coalition array has *property* II if, for all $q \in \{1, \dots, p\}$ there is a system of coefficients $c_S > 0$ ($S \in \sum_{j=1}^{q} \underline{\underline{B}}_j$) and $c_S \geqslant 0$ ($S \in \underline{\underline{B}}_0$) such that

$$\sum_{S \in \bigcup_{j=0}^{q} \underline{\underline{B}}_j} c_S 1_S = 1_{\Omega}. \tag{14}$$

This means in particular that $\bigcup_{j=0}^{q} B_j$ is weakly balanced.

The above exposition follows Kohlberg [4]. For our purpose we quote some of his results as follows.

Theorem 2.3 [4]. *If v is a homogeneous game, then*

$$\mathcal{N}(v) = \{x \in \mathcal{X} \mid \mathcal{B}(x, v) \text{ has property I}\}$$
$$= \{x \in \mathcal{X} \mid \mathcal{B}(x, v) \text{ has property II}\}.$$

Again, it should be noted that the assumption $v(\{i\}) = 0$ ($i \in \Omega$) (i.e., there are no winning players) can be dropped without destroying the proofs.

Theorem 2.4. *Let v be a homogeneous game and let*

$$\kappa = \max\{i \in \Omega \mid \{i\} \in \underline{\underline{W}}^m\}$$

be the smallest winning player ($\max \emptyset = a - 1$).

(1) *If $\kappa = b$ then*

$$\nu(v) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right) \quad (n := b - a + 1).$$

(2) *If $\kappa < b$, let $\tilde{v}$ denote the homogeneous game on $\tilde{\Omega} = [\kappa + 1, b]$ which is obtained by dropping the winning players. Also, let $\tilde{\nu} = \nu(\tilde{v})$ and*

$$\tilde{\alpha} := 1 - \mu(\tilde{\nu}, \tilde{v}),$$

$$\tilde{x} := (\underbrace{\tilde{\alpha}, \dots, \tilde{\alpha}}_{\kappa + 1 - a \text{ times}}, \tilde{\nu}_{k+1}, \dots, \tilde{\nu}_b)/((k + 1 - a)\tilde{\alpha} + 1). \tag{15}$$

Then $\nu(v) = \tilde{x}$.

In other words, the pseudo-nucleolus of v is obtained by computing the one on $\tilde{v}$, then assigning $\tilde{\alpha}$ to the winning players and finally rescaling.

Proof. The cases $\kappa = b$ and $\kappa = a - 1$ are trivial; so we have to concentrate on the second case for $a \leqslant \kappa < b$.

Consider the coalition array $\mathscr{B}(\tilde{x}, v)$, we would like to show that it enjoys property I. To this end, fix $q \in [1, p]$ and let $y \in \mathbb{R}^{\Omega}$ satisfy (12) and (13).

First of all note that $y_j \geqslant 0$ for $j \in [a, \kappa]$. For, in view of $e(\tilde{x}, \{j\}) = 1 - \tilde{x}_j = 1 - (1 - \mu(\tilde{v}, \tilde{v}))/((\kappa + 1 - a)\tilde{\alpha} + 1)$, it turns out that $\{j\} \in \underline{\underline{B}}_1(\tilde{x}, v)$; Thus, y is nonnegative on $[a, \kappa]$.

Next, if $y([a, \kappa]) = 0$, then clearly $y(S) = 0$ for all $S \in \sum_{j=1}^{q} \underline{\underline{B}}_j(\tilde{x}, v)$; this follows by the fact that $\tilde{v}$ is the nucleolus of $\tilde{v}$. If $\tilde{v}$ is the unanimity game then obviously $y([k + 1, b]) \geqslant 0$, thus $y([a, k]) \leqslant 0$; therefore we assume that $\tilde{v}$ is not the unanimity game. In this case, if $0 < y([a, \kappa]) =: \tilde{\beta}$, then define

$$\tilde{y} := \left(y_{\kappa+1} + \frac{\tilde{\beta}}{b - \kappa}, \ldots, y_b + \frac{\tilde{\beta}}{b - \kappa} \right). \tag{16}$$

It is not hard to see that $\tilde{y}$ indeed satisfies (12) and (13) with respect to the game $\tilde{v}$ and, say, $q = 1$. This is a contradiction to Theorem 2.3, since $\tilde{v}$ is the nucleolus of $\tilde{v}$, hence this case cannot occur and we have finished our proof. $\square$

The last theorem shows that we may disregard the case that winning players are present.

Hence, from now on we shall assume that all homogeneous games under consideration have no winning players (i.e., $M_i < \lambda$ $(i \in \Omega)$ for any representation (M, λ) of some v).

Consequently, the prefixes "pre" and "pseudo" may be omitted, thus $\mathscr{X}$ is the set of imputations and $v = v(v)$ the nucleolus.

Remark 2.5 (Kohlberg [4, Theorem 1.4]). Let κ denote for the moment the last player who gets a payoff with the nucleolus, i.e.,

$$\kappa = \max\{i \in \Omega \mid v_i > 0\}. \tag{17}$$

Then $\{S \cap [a, \kappa] \mid S \in \mathscr{M}(x, v)\}$ is strongly balanced.

If v is a homogeneous game "with steps of different type" (other than the smallest nondummy), then it can be inferred easily, that $\underline{\underline{W}}^m$ cannot be strongly balanced (see also Remark 5.4 in Peleg and Rosenmüller [9]).

Now, in view of the exhibition presented in Section 1, we can easily show that $\underline{\underline{W}}^m$ cannot be weakly balanced. In fact, we show a bit more:

Corollary 2.6. *Let v be a homogeneous game with steps of different type (no dummies, no winning players). Then 1_{Ω} is no linear combination of $(1_S)_{S \in \underline{\underline{W}}^m}$.*

Proof. Let $(M(\cdot), \lambda(\cdot))$ be a *strictly* monotone a.f.r. and suppose that, for some system of coefficients $(c_S)_{S \in \underline{\underline{W}}^m}$, we have

$$\sum_{S \in \underline{\underline{W}}^m} c_S 1_S = 1_{\Omega}.$$

Then

$$M(a)(\Omega) = \sum_{S \in \underline{\underline{W}}^m} c_S M(a)(S) = \lambda(a) \sum_{S \in \underline{\underline{W}}^m} c_S. \tag{18}$$

Now, $\sum c_S$ is a constant, thus (18) contradicts Theorem 1.11, which states that the quotient $\lambda(a)/(M(a)(\Omega))$ is a strictly increasing function. □

3. The nucleolus for games with steps

As we have mentioned, we will from now on only deal with homogeneous games without dummies and winning players.

The behavior of the nucleolus for games "without steps" has been described in Peleg and Rosenmüller [9]. Here, we want to tackle the same problem when steps are present.

There is an inductive procedure involved in our method which (unlike the method of satellite games as explained in Section 0) uses a truncation procedure cutting off smaller players. To explain this version of "truncated games" we have to shortly recall the theory of the *incidence vector* of a homogeneous game (without steps), as developed by Sudhölter [14].

To this end we fix $\Omega = [1, n]$ (!) *throughout this section* and focus our attention (*initially only*) on a homogeneous game v *without* steps. Let (M, λ) be its unique minimal representation so that $v = v_\lambda^M$.

Next, recall the replacement procedures as described in Section 1: Player $j \in S \in \underline{\underline{W}}^m$ is expendable in S, if

$$[j, l] \subseteq S, \qquad S - j + [l + 1, n] \in \underline{\underline{W}}, \tag{1}$$

where $l = l(S)$; replacing him yields a coalition

$$\rho_j(S) := S - j + [l(S) + 1, t] \tag{2}$$

with suitable $t \in [l(S) + 1, n]$. And if $r \notin T \in \underline{\underline{W}}^m$, such that

$$[r + 1, l(T)] \subseteq T \tag{3}$$

then r is the last dropout; inserting him yields

$$\varphi(T) := T + r - [t', l(T)]. \tag{4}$$

The relations

$$\rho_r(\varphi(T)) = T \tag{5}$$

and

$$\varphi(\rho_j(S)) = S \tag{6}$$

are immediate consequences (cf. Section 1).

According to Sudhölter [14] we have

Lemma 3.1 (cf. [14, Theorem 2.3, Definition 2.4]). *Let v be a homogeneous game (without dummies and winning players). Assume that v has no steps. Then there is a unique sequence $S_1, \ldots, S_n$ of min-win coalitions defined by the following procedure.*

(1) $S_1 = S^{(\lambda)}$.

(2) *For every* $k \in [1, n-1]$, *the system* $\underline{\underline{S}}_k := \{S_i \mid i \in [1,k],\ k \text{ is expendable in } S_i\}$ *is nonempty.*

(3) *Among all* $S_i \in \underline{\underline{S}}_k$ *with minimal length* $l(S_i)$, *let* S_{i_0} *be the one with smallest (first) index.*

(4) $S_{k+1} = \rho_k(S_{i_0})$.

Definition 3.2. Let $S_1, \ldots, S_n$ be given by Lemma 3.1. Then

$$l = l^{(v)} = (l_1, \ldots, l_n) := (l(S_1), \ldots, l(S_n))$$

is the *incidence vector* of v.

The incidence vector characterizes v uniquely [14, Theorem 2.10]. (The term "incidence vector" can be defined abstractly.)

Given the incidence vector, the game v can be obtained by "reversing" the procedure of Lemma 3.1. In other words, the sequence $S_1, \ldots, S_n$ can be constructed in a unique way and, since we are dealing with a game without steps, the unique minimal representation is obtained at once.

Let us shortly describe this "reversal procedure".

Given $l = l^{(v)} = (l_1, \ldots, l_n)$, the *staircase* corresponding to $l^{(v)}$ (and hence to v) is the vector

$$\pi = \pi(v) \in \mathbb{N}^n$$

given by

$$\pi_k = \min\{l_j \mid j \leqslant k \leqslant l_j\} \quad (k = 1, \ldots, n) \tag{7}$$

(with the convention that $\min \emptyset = 0$). If π is regarded as a function of k, then it is monotone and can be identified as a "quadratic step function" since the heights of jumps and the length of plateaus are equal (see [14, Section 3]). E.g., if l equals

$$l = (3, 7, 6, 5, 7, 7, 7, 8) \tag{8}$$

then

$$\pi = (3, 3, 3, 5, 5, 6, 7, 8).$$

Thus π appears as a staircase with square steps that vary in height and width simultaneously (and with appropriate view, l decreases on the plateaus of π but dominates π, cf. Fig. 3).

(On the other hand, π_k denotes of course the minimal length of a coalition in $\underline{\underline{S}}_k$, if we view Lemma 3.1.)

Now, define the *selector* to be the vector $\omega = \omega^{l(v)} = \omega^{(v)}$ which is given by

$$\omega_k = \min\{j \mid l_j = \pi_k\} \tag{9}$$

(again $\min \emptyset = 0$). Then ω selects the appropriate index i_0 in the formulation of Lemma 3.1. More precisely, given l, the sequence $S_1, \ldots, S_n$ as specified by Lemma 3.1 is given by

$$S_1 = [1, l_1], \tag{10.1}$$

$$S_{k+1} = S_{\omega_k} - k + [l_{\omega_k} + 1, l_{k+1}] = \rho_k(S_{\omega_k}). \tag{10.2}$$

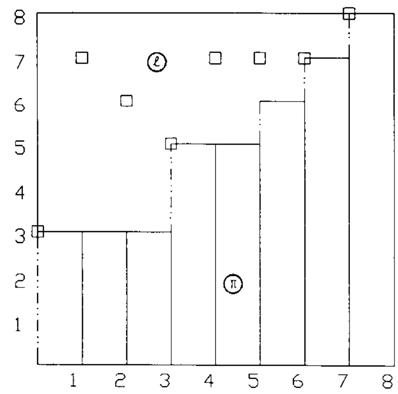

Fig. 3.

E.g., in the example suggested by (8), we obtain

$$\omega = (1, 1, 1, 4, 4, 3, 2, 8)$$

telling us that, e.g., in the fifth step of the construction suggested by Lemma 3.1 we have to render player 4 to become the last dropout in S_4.

Remark 3.3 (Sudhölter [14]). Let (M, λ) be the minimal representation of a homogeneous game *with* steps; assume $\Omega = [1, n]$ and write $M = (M_1, \ldots, M_n)$. Let $\hat{M} := (M_1, \ldots, M_n, 1)$. Then $v_\lambda^{\hat{M}}$ is an ($(n+1)$-person) homogeneous game *without* steps and $(\hat{M}, \lambda)$ is its minimal representation.

Intuitively, if we add a player of weight 1, then his weight can just be used to close the "jumps" that appear at a step (cf. Section 0).

Definition 3.4. Let $\Omega = [1, n+1]$ and let l be an incidence vector (of a hom game for $n + 1$ players *without* steps). Let $\kappa \in [2, n]$. The *truncation of l at κ* is the vector $\mathring{l} = \mathring{l}^{(\kappa)} \in \mathbb{N}^{\kappa+1}$ given by

$$\mathring{l}_i = \begin{cases} l_i, & l_i \leqslant \kappa, \\ \kappa, & \kappa < l_i \leqslant n,\ \pi_{i-1} < \kappa, \\ \kappa + 1, & \text{otherwise.} \end{cases} \tag{11}$$

($\pi \in \mathbb{N}^{n+1}$ is the staircase corresponding to l!)

Remark 3.5. $\mathring{l}^{(\kappa)}$ is an incidence vector.

Proof. This follows immediately by the observation that $\mathring{l}^{(\kappa)}$ enjoys a corresponding staircase, namely

$$\mathring{\pi}^{(\kappa)} = (\kappa \wedge \pi|_{[1,\kappa]}, \kappa + 1) \tag{12}$$

where "min" ($= \wedge$) has to be taken coordinatewise. □

Definition 3.6. Let v be a homogeneous game (with steps) on $\Omega = [1,n]$. Let $\kappa \in [2,n]$. The *truncation of* v *at* κ, $\tilde{v}^{(\kappa)}$ is defined as follows.

(1) Let (M,λ) be the minimal representation. Let $(\hat{M},\lambda)$ be obtained by Remark 3.3 and let $\hat{l}$ be the incidence vector of $\hat{v} = v_\lambda^{\hat{M}}$.

(2) Let $\mathring{l} \in \mathbb{N}^{\kappa+1}$ be the truncation of $\hat{l}$ at κ as defined by Definition 3.4. $\mathring{l}$ generates a homogeneous game $\mathring{v}$ on $[1,\kappa+1]$ with minimal representation $(\mathring{M},\mathring{\lambda})$, $\mathring{M} \in \mathbb{N}^{\kappa+1}$.

(3) $\tilde{v}^{(\kappa)}$ is the game which is (minimally) represented by

$$\tilde{M} := (\mathring{M}_1,\ldots,\mathring{M}_\kappa) = \mathring{M}|_{[1,\kappa]},$$

$$\tilde{\lambda} := \mathring{\lambda}.$$

Note that homogeneous games without steps indeed attach weight 1 to the smallest *two* players (w.r.t. the minimal representation). Of course the one-to-one correspondence between homogeneous games and incidence vectors is heavily used (cf. Theorem 2.10 of Sudhölter [14]).

Our first aim is to obtain some insight into the structure of the truncated versions. The following lemma is an attempt to describe the min-win coalitions of some $\tilde{v}^{(\kappa)}$.

To this end, let us slightly augment our notation:

If $S \in \underline{\underline{W}}^m(v)$ and $r \notin S$, $r < l(S)$ (r is *any* dropout), then:

$$\varphi_r(S) := (S \cap [1,r-1]) + [r,t'] \in \underline{\underline{W}}^m(v). \tag{13}$$

Thus, $\varphi_r(S)$ is the lexicographical first min-win coalition among all coalitions T with $T \cap [1,r-1] = S \cap [1,r-1]$.

Lemma 3.7. *Let* v *be a homogeneous game (on* $\Omega = [1,n]$*) and let* $\kappa \in [2,n]$*. Then* $\tilde{v}^{(\kappa)}$ *has the following properties.*

(1) *If* $i \in [1,\kappa]$ *is a step of* $\tilde{v}^{(\kappa)}$*, then* i *is a step of* v *or* $i \sim_{\tilde{v}^{(\kappa)}} \kappa$*. On the other hand, if* $i \in [1,\kappa]$ *is a step of* v*, then he is a step of* $v^{(\kappa)}$.

(2) *If* $S \subseteq [1,\kappa]$ *and* $S \in \underline{\underline{W}}^m(v)$*, then* $S \in \underline{\underline{W}}^m(\tilde{v}^{(\kappa)})$.

(3) *If* $S \subseteq [1,\kappa]$ *and* $S \in \underline{\underline{W}}^m(\tilde{v}^{(\kappa)}) - \underline{\underline{W}}^m(v)$*, then* $\kappa \in S$ *and* $S \notin \underline{\underline{W}}(v)$.

(4) *If* $S \subseteq [1,\kappa]$ *and* $S \in \underline{\underline{W}}(\tilde{v}^{(\kappa)})$*, then* $S + [\kappa+1,n] \in \underline{\underline{W}}(v)$.

(5) *If* $S \supseteq [1,\kappa]$ *and* $S \in \underline{\underline{W}}^m(v)$*, then* $[1,\kappa] \in \underline{\underline{W}}^m(\tilde{v}^{(\kappa)})$ *and* $\tilde{v}^{(\kappa)}$ *is the unanimous game of* $[1,\kappa]$.

(6) *If* $S \in \underline{\underline{W}}^m(v)$, $S \subseteq\!\!\!| \, [1,\kappa]$, $S \supseteq\!\!\!| \, [1,\kappa]$, *and, with* $r = l([1,\kappa] \cap S^c)$, $\varphi_r(S) \subseteq [1,\kappa-1]$*, then* $S \cap [1,\kappa] \in \underline{\underline{W}}^m(\tilde{v}^{(\kappa)})$.

Proof. (1) Given v, let $\mathring{l} \in \mathbb{N}^{\kappa+1}$ be defined via Definition 3.6(2). In view of [14, Chapter 2], it is known that player i is a step w.r.t. $\tilde{v}^{(\kappa)}$ iff $\mathring{l}_{i+1} = \kappa + 1$. In view of Definition 3.4, this leaves two alternatives for l_{i+1}: either $l_{i+1} = n+1$, in which case i is a step of v. Or else $\pi_i \geqslant \kappa$. But then (see (12)) $\pi_i^{(\kappa)} = \kappa$ and $i \sim_{\tilde{v}^{(\kappa)}} \kappa$.

The reverse statement is seen analogously.

(2) To prove the second statement, assume that, on the contrary, for some $\bar{S} \subseteq [1,\kappa]$ it turns out that $\bar{S} \in \underline{\underline{W}}^m(v)$ and $\bar{S} \notin \underline{\underline{W}}^m(\tilde{v}^{(\kappa)})$.

Clearly, since in this case $l_1 = \mathring{l}_1$, $\bar{S}$ is not the lex-max min-win coalition of v. Hence, there exists the last dropout of $\bar{S}$, say $r \notin \bar{S}$, $r < l(S)$.

Now, among all $\bar{S}$ with this property collect those with minimal length $l(\bar{S})$. And, among all those with minimal length, choose a one with maximal last dropout r. Call these now $\bar{S}$ and r again.

Define

$$T := \varphi(\bar{S}). \tag{14}$$

Then, because $l(T) < l(\bar{S})$ holds true, it follows from our choice of $\bar{S}$ that

$$T \in \underline{\underline{W}}^m(v) \cap \underline{\underline{W}}^m(\tilde{v}^{(\kappa)}). \tag{15}$$

Next, we know that the procedure indicated by Lemma 3.1 (and [14, Theorem 2.3]) yields two min-win coalitions of v, say S_{i_0} and S_{r+1} such that player r is expendable in S_{i_0} and

$$S_{r+1} = \rho_r(S_{i_0}), \qquad \varphi(S_{r+1}) = S_{i_0}. \tag{16}$$

More precisely,

$$l(S_{i_0}) = \min\{l(S) \mid S \ni r,\ S \in \underline{\underline{W}}^m(v)\} = \pi_r \tag{17}$$

and

$$l(S_{r+1}) = \min\{l(S) \mid S \not\ni r,\ S \ni r+1,\ S \in \underline{\underline{W}}^m(v)\} = l_{r+1} \tag{18}$$

while

$$S_{i_0} \cap [1, r-1] = S_{r+1} \cap [1, r-1] \tag{19}$$

is also true. Clearly, $S_{i_0} \in \underline{\underline{W}}^m(\tilde{v}^{(\kappa)})$ in view of (17); in fact it follows from (17) that $l(S_{i_0}) \leqslant l(T)$. However, $l(S_{i_0}) < l(T)$ is impossible in view of our choice of $\bar{S}$ and r. But $l(S_{i_0}) = l(T)$ implies via application of φ (cf. (14) and (16)) that $l(S_{r+1}) = l(\bar{S})$ holds true.

In this case, (18) shows that $\bar{S} \in \underline{\underline{W}}^m(\tilde{v}^{(\kappa)})$, and we have completed our proof of the second statement.

(3) The third statement is verified by a sequence of analogous arguments.

(4) Follows from the definition of $\mathring{l}$.

(5) A trivial consequence.

(6) Follows from Definition 3.6 and from (1). □

We are now in the position to tackle the nucleolus of a homogeneous game with steps. To this end, in what follows $\underline{\tau} = \underline{\tau}(v)$ denotes the first (largest) step of a homogeneous game v, i.e.,

$$\underline{\tau} = \min\{i \in \Omega \mid i \in \Pi(v)\}. \tag{20}$$

Similarly, $\tau = \tau(v)$ is the smallest player of the type of $\underline{\tau}$, i.e.,

$$\tau = \max\{i \in \Omega \mid i \underset{v}{\sim} \underline{\tau}\}. \tag{21}$$

Note that $[\underline{\tau}, \tau]$ consists of steps that appear as block in any min-win coalition if they appear at all. Of course $\underline{\tau} = \tau$ will frequently happen.

Theorem 3.8. *Let v be a homogeneous game on $\Omega = [1, n]$ and let $\tau = \tau(v)$ be the smallest player of the largest step's type. Let $\nu = \nu(v)$ be the nucleolus of v. Then $\nu_{\tau+1} = \cdots = \nu_n = 0$.*

Clearly, we have to treat the case of a game v with steps of different type only. Then, Corollary 2.6 shows at once that $v_n = 0$ is necessarily true. The problem is that Corollary 2.6 cannot be employed immediately in order to prove that all players behind the first steps get zero at the nucleolus, here we have to fall back on a truncation.

Proof. Assume that, on the contrary, there is $\kappa \in [\tau + 1, n]$ such that

$$v_{\tau+1} \geqslant \cdots \geqslant v_\kappa > 0 = v_{\kappa+1} = \cdots = v_n.$$

The proof proceeds by treating various cases separately.

Case 1: *Assume that there is a coalition $\bar{S} \in \mathscr{M}(v, v)$ such that $\tau \in \bar{S}$, $[\tau + 1, \kappa] \subseteq |\bar{S}$.*

This case is of course easy: define an imputation $x \in \mathscr{X}$ via

$$x_i = \begin{cases} v_i, & i \in [1, \tau - 1] \cup [\kappa + 1, n], \\ v_\tau + \dfrac{1}{2} \displaystyle\sum_{j=\tau+1}^{\kappa} v_j, & i = \tau, \\ \dfrac{v_i}{2}, & i \in [\tau + 1, \kappa], \end{cases} \tag{22}$$

and observe that $\mu(x, v) \leqslant \mu(v, v)$ while $\bar{S} \notin \mathscr{M}(x, v)$.

Because "steps rule their followers", no min-win coalition has larger excess at x than at v and $\bar{S}$ has a smaller one, this contradicts the fact that v is the nucleolus. This finishes our proof for Case 1 at once.

We may now assume that no $\bar{S}$ of the kind treated already exists.

Then $\mathscr{M} = \mathscr{M}(v, v)$ allows for a partition, say

$$\mathscr{M} = \mathscr{M}_- + \mathscr{M}_+ \tag{23}$$

such that

$$\begin{aligned} \mathscr{M}_- &:= \{S \in \mathscr{M} \mid [\tau, \kappa] \subseteq S^c\}, \\ \mathscr{M}_+ &:= \{S \in \mathscr{M} \mid [\tau, \kappa] \subseteq S\} \end{aligned} \tag{24}$$

holds true.

Both sets are nonempty since the nucleolus of a game is contained in the kernel [12].

Now, we turn to the truncation $\tilde{v}^{(\kappa)}$ of v at κ which, for short, we abbreviate by $\tilde{v} := \tilde{v}^{(\kappa)}$.

By Lemma 3.7, we know that τ is a step w.r.t. $\tilde{v}$ and, "in $\tilde{v}$", τ may or may not be of the same type as κ (see Lemma 3.7(1)).

Accordingly, the next two cases treat these two possibilities. The easier one, in which τ and κ are of different type, is considered first.

Case 2: *Let us treat the case that $\tau \not\sim_{\tilde{v}} \kappa$.*

This means in fact that τ is the smallest player of the largest step's type *also in* $\tilde{v}$ (see Lemma 3.7(1)).

First of all, let us define a mapping

$$* : \mathscr{M} \to \underline{\underline{W}}^m(v) \tag{25}$$

separately for $S \in \mathscr{M}_-$ and $S \in \mathscr{M}_+$.

(1) For $S \in \mathscr{M}_-$, define

$$S^* := S \cap [1, \tau]. \tag{26}$$

Indeed, $S^* \in \underline{\underline{W}}^m(v)$ is true since "step τ rules his followers", thus the smaller players to the right of τ (of κ, since $S \in \mathscr{M}_-$) cannot appear in a min-win coalition without τ. Hence a min-win coalition has to be contained in S^*. It cannot be properly contained since $S \in \mathscr{M}$.

(2) For $S \in \mathscr{M}_+$, define

S^* is the lexicographically first coalition in $\underline{\underline{W}}^m(v)$ which satisfies $S^* \cap [1, \tau] = S \cap [1, \tau]$. (27)

Because of the decomposition (23) and (24), S^* cannot cut into $[\tau, \kappa]$, thus $l(S^*) \geqslant \kappa$ and $[\tau, l(S^*)] \subseteq S^*$.

From our definition of S^* we conclude:

$$S^* \cap [1, \kappa] \in \underline{\underline{W}}^m(\tilde{v}). \tag{28}$$

Indeed, for $S \in \mathscr{M}_-$ this follows from Lemma 3.7(2), and for $S \in \mathscr{M}_+$ this follows from Lemma 3.7(6).

Furthermore, it is seen that

$$S \cap [1, \kappa] = S^* \cap [1, \kappa] \tag{29}$$

for all $S \in \mathscr{M}$.

The final conclusion is straightforward: By Kohlberg's result (Remark 2.5) we obtain a set of nonnegative real numbers $(c_S)_{S \in \mathscr{M}}$ such that

$$\sum_{S \in \mathscr{M}} c_S 1_{S \cap [1,\kappa]} = 1_{[1,\kappa]}. \tag{30}$$

By (29),

$$\sum_{S \in \mathscr{M}} c_S 1_{S^* \cap [1,\kappa]} = 1_{[1,\kappa]}. \tag{31}$$

This, in view of (28), means that $1_{[1,\kappa]}$ ($[1,\kappa]$ is the grand coalition in $\tilde{v}$!) is a linear combination of $(1_S)_{S \in \underline{\underline{W}}^m(\tilde{v})}$. Since $\tilde{v}$ has steps (at least τ), this contradicts Corollary 2.6.

Case 3: *Now we treat the case that* $\tau \sim_{\tilde{v}} \kappa$.

Again, we want to construct some contradiction between Remark 2.5 and Corollary 2.6; however, as we are not in the position to claim that $\tilde{v}$ has steps, the procedure of the case has to be modified. We will eventually consider the truncation $v' = \tilde{v}^{(\kappa'+1)}$ for some $\kappa' \geqslant \kappa$ and in this truncation (30) and (31) will have appropriate analogues.

To this end, let us proceed by several steps. The first is to define the "critical player" κ'.

Step 1. As τ is the smallest player of his type "in v", there is $\bar{T} \in \underline{\underline{W}}^m(v)$ such that $\tau \in \bar{T}$, $\tau + 1 \notin \bar{T}$.

Since $\tau \sim_{\tilde{v}} \kappa$, it follows necessarily that $\bar{T} \cap [\kappa + 1, n] \neq \emptyset$, otherwise $\bar{T}$ would be min-win in $\tilde{v}$ (Lemma 3.7(2)) and separate τ and $\tau + 1$.

Now, choose $\bar{T}$ to be lexicographically maximal with the above properties (i.e., $\tau \in \bar{T}$, $\tau + 1 \notin \bar{T}$, $\bar{T} \cap [\kappa + 1, n] \neq \emptyset$, $T \in \underline{\underline{W}}^m(v)$). Then we have, in addition

$$[\tau + 2, l(\bar{T})] \subseteq \bar{T} \tag{32}$$

and

$$l(\bar{T}) > \kappa. \tag{33}$$

Again, among all coalitions with these properties, choose the one with minimal length. Define

$$\kappa' := l(\varphi_{\tau+1}(\bar{T})) = \min\{l(S) \mid S \in \underline{\underline{W}}^m(v),\ \tau + 1 \in S\}. \tag{34}$$

Now, $\kappa' \geqslant \kappa$ holds true. Indeed, otherwise $\tau \sim_{\tilde{v}} \kappa$ would be violated by Lemma 3.7(6).

Consider now the truncation of v at $\kappa' + 1$, say

$$v' = \tilde{v}^{(\kappa' + 1)}.$$

By Lemma 3.7(6) and (34) it follows that $\bar{T} \cap [1, \kappa' + 1] \in \underline{\underline{W}}^m(v')$, and as $\tau + 1 \notin \bar{T}$, τ and $\tau + 1$ are of different type in v'.

Thus (see Lemma 3.7(1)) it turns out that τ is the smallest player of the largest steps type w.r.t. v', i.e.,

$$\tau(v') = \tau(v) = \tau.$$

Therefore, we shall now concentrate our efforts on v' and try to imitate the procedure of Case 2.

Step 2. Consider any coalition $\hat{S} \in \mathscr{M}_+$ such that $[\kappa + 1, l(\hat{S})] \subseteq \hat{S}$. Such coalitions exist: we may generate them from arbitrary elements of $\mathscr{M}_+$ by successively involving the last dropout. We claim:

$$l(\hat{S}) > \kappa' \quad \text{for all } \hat{S} \text{ with } \hat{S} \in \mathscr{M}_+,\ [\kappa + 1, l(\hat{S})] \subseteq \hat{S}. \tag{35}$$

Indeed, if for some $\hat{S}$, (35) is violated, then consider

$$\hat{\hat{S}} := \hat{S} - \{\tau + 1\} + \{\kappa' + 1, \ldots, l(\bar{T})\}.$$

This is a winning coalition of v which satisfies

$$v(\hat{\hat{S}}) = v(\hat{S}) - v_{\tau+1} < v(\hat{S}) < 1 - \mu(v, v),$$

contradicting the fact that v is the nucleolus of v.

Step 3. We can now repeat our argument, as presented in Case 2, but for v'.

Again define $* : \mathscr{M}(v, v) \to \underline{\underline{W}}^m(v)$.

For $S \in \mathscr{M}_-$:

$$S^* := S \cap [1, \tau] \tag{36}$$

and $S^* \in \underline{\underline{W}}^m(v)$ follows exactly as in Case 2, while $S^* \cap [1, \kappa' + 1] = S \cap [1, \kappa' + 1]$ is trivial.

For $S \in \mathscr{M}_+$, choose $S^* \in \underline{\underline{W}}^m(v)$ to be lexicographically maximal with

$$S^* \cap [1, \tau] = S \cap [1, \tau]. \tag{37}$$

Then $S^* \cap [\kappa' + 1] = S \cap [\kappa' + 1]$ follows from (35). Again, in view of Lemma 3.7(2) and (6),

$$\{S^* \cap [1, \kappa' + 1] \mid S \in \mathscr{M}\} \subseteq \underline{\underline{W}}^m(v'). \tag{38}$$

Next, by Remark 2.5, we find coefficients $(c_S)_{S \in \mathscr{M}}$ such that

$$\sum_{S \in \mathscr{M}} c_S 1_{S \cap [1, \kappa]} = 1_{[1, \kappa]}. \tag{39}$$

But for $S \in \mathscr{M}_-$ it is clear that $S^* \cap [1, \kappa' + 1] = S \cap [1, \kappa]$ (cf. (36)).

Fortunately, for $S \in \mathscr{M}_+$, (35) and (37) yield $S^* \cap [1, \kappa' + 1] \supseteq [\kappa + 1, \kappa' + 1]$. Thus, from (39) it follows that

$$\sum_{S \in \mathscr{M}} c_S 1_{S^* \cap [1, \kappa' + 1]} = 1_{[1, \kappa' + 1]}. \tag{40}$$

But, in view of (38), (40) contradicts Corollary 2.6. □

Corollary 3.9. *With the notation of Theorem* 3.8 *the vector*

$$(\nu(\tilde{v}^{(\tau)}), \underbrace{0, \ldots, 0}_{n - \tau \text{ times}})$$

is the nucleolus of v.

Proof. By Theorem 3.8, $\nu_j = 0$ for all $j > \tau$. Thus ν is the nucleolus of the game $v^M_{\lambda - M([\tau + 1, n])}$ (where $v = v^M_\lambda$). This game obviously coincides with the truncated game $\tilde{v}^{(\tau)}$ with $n - \tau$ additional dummies. □

Example 3.10. Consider the pair

$$(M, \lambda) = (12, 10, 5, 3, 2, 2, 1, 1; 22)$$

and the game $v = v^M_\lambda$. Put $a = 1$, $b = 8$, i.e., $\Omega = \{1, \ldots, 8\}$. Then it is easy to see that players 3 and 8 are the steps of v. Thus, according to Theorem 3.8, we have $\tau(v) = 3$. Writing coalitions as 0-1 vectors is instructive, thus from the following sequence of min-win coalitions (cf. Lemma 3.1)

$$
\begin{aligned}
S_1 &= (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\\
S_2 &= (0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0)\\
S_3 &= (1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0)\\
S_4 &= (1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1)\\
S_5 &= (1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0)\\
S_6 &= (1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0)\\
S_7 &= (0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0)\\
S_8 &= (1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0)\\
S_9 &= (0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1)
\end{aligned}
$$

The coalition S_3 at the origin of this arrow is used to construct the coalition S_6 at the top of the arrow via $\rho_5(S_3)$.

we obtain the incidence vector $\hat{l}$ of $\hat{v} = v_{\lambda}^{\hat{M}}$ (cf. Definition 3.6):

$$\hat{l} = (2, 6, 5, 9, 7, 6, 8, 8, 9).$$

Now the truncation of $\hat{l}$ at 3 is $(2, 3, 3, 4)$, which generates $(1, 1, 1, 1; 2)$, thus the truncated game $\tilde{v}^{(3)}$ can be represented by $(1, 1, 1; 2)$, showing that all players are of the same type. Corollary 3.9 at once enables us to write down the nucleolus of v:

$$(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0, 0, 0).$$

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